# Relativistic fixed-energy amplitudes of the step and square well potential problems



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# Relativistic fixed-energy amplitudes of the step and square well potential problems

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**Abstract.** The fixed-energy amplitudes (Green's function) of the one-dimensional relativistic Wood–Saxon, step and square well potential problems are calculated with the help of Kleinert's path integral technique for relativistic potential problems.

## 1. Introduction

During the last 15 years, considerable progress has been made in solving path integrals of potential problems [1, 2]. It is no exaggeration to say that practically all problems in quantum mechanics solved exactly by the Schrödinger equation can now be done by path integral (PI). But, the same thing cannot be said for relativistic potential problems.

Particles moving at large velocities near the speed of light are called relativistic particles. If such particles interact with each other or with an external potential, they exhibit quantum effects which cannot be described by their orbital fluctuations alone. At very short interaction times, additional particles or pairs of particles and antiparticles are created or annihilated, and the total number of particle orbits is no longer invariant. Ordinary quantum mechanics which always assumes a fixed number of particles cannot describe such processes. The associated path integral has the same problem since it is a sum over a given set of particle orbits. Thus, even if relativistic kinematics is properly incorporated, a path integral cannot yield an accurate description of relativistic particles. An extension becomes necessary which includes an arbitrary number of mutually linked and branching fluctuating orbits.

Fortunately, a more efficient way of dealing with relativistic particles exists. It is provided by quantum field theory (see, e.g. [3, 4]). The branch points of newly created particle lines are considered by anharmonic terms in the field action. The calculation of their effects proceeds by the perturbation theory performed systematically in terms of Feynman diagrams. They consist of lines and vertices representing direct pictures of the various interconnections of particle orbits.

Nevertheless, from the historical point of view, the attempt at finding a relativistic generalization of the Schrödinger equation is an important step towards the development of quantum field theory. For this reason, many textbooks on quantum field theory begin with a discussion on relativistic quantum mechanics. So far, only a few relativistic problems have been discussed by PI (those in [1, 5–7]). In this paper, we shall solve the other relativistic problems by path integral. The treatment will be restricted to spinless particles. Problems

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whose PIs will be calculated are: (1) a one-dimensional relativistic particle moving in the Wood–Saxon potential, (2) a relativistic particle moving in the step potential and (3) a relativistic particle moving in the one-dimensional square well potential. In view of non-relativistic solutions of these problems, details can be found in many textbooks. The potential in (1) with an impenetrable wall at $x = 0$ has been widely used for the optical potential and has been very successful at reproducing experimental scattering cross sections for different projectiles (see, e.g. [8]). System (2) is a favourite model for explaining a number of basic quantum effects. Model (3) has been used to describe approximately phenomena caused by $\pi^+$–$\pi^-$ pairs in the nucleus [9]. During the last 15 years, the PIs of these systems in non-relativistic case have been solved [1, 10–14].

This paper is organized as follows. In section 2, we briefly review the formulation of the PI for the relativistic potential problems. In section 3, with the help of the formuation in section 1, we present the fixed-energy amplitude of the relativistic Wood–Saxon potential problem. Furthermore, we solve the relativistic step and square well problems by suitable limits and auxiliary $\delta$-function perturbation. Our conclusions are summarized in section 4.

## 2. Path integrals for relativistic particle orbits

To obtain a PI representation of the Green's function of the Klein–Gorden field, we use the following action [6]

$$A = \int_{\lambda_a}^{\lambda_b} \mathrm{d}\lambda \left[\frac{M}{2\rho(\lambda)} x'^2(\lambda) + \frac{Mc^2}{2}\rho(\lambda)\right] \tag{1}$$

where $\rho(\lambda)$ is an extra dimensionless fluctuating variable and $x = (\boldsymbol{x}, \tau)$ is a $(D+1)$ vector with the invariant length $x = \sqrt{\boldsymbol{x}^2 + c^2\tau^2}$, i.e. the metric has the form

$$(g_{\mu\nu}) = \mathrm{diag}(1, \ldots, 1, c^2). \tag{2}$$

The action of equation (1) coincides with the classical action

$$A_{cl} = Mc \int_{\lambda_a}^{\lambda_b} \mathrm{d}\lambda \sqrt{x'^2(\lambda)} \tag{3}$$

since equation (1) is extremal for

$$\rho(\lambda) = \frac{1}{c}\sqrt{x'^2(\lambda)}. \tag{4}$$

Inserting equation (4) back into equation (1) yields the classical action of equation (3). The new action $A$ shares the reparametrization invariance

$$\lambda \longrightarrow f(\lambda) \tag{5}$$

with the classical action $A_{cl}$ if $\rho(\lambda)$ is simultaneously changed as

$$\rho(\lambda) \longrightarrow \rho(\lambda)/f'(\lambda). \tag{6}$$

The action $A$ has the advantage of being quadratic in the orbital variable $x(\lambda)$. Because the physical time is analytically continued to imaginary values so that the metric becomes Euclidean, the action looks like that of a non-relativistic particle moving as a function of a pseudotime $\lambda$ through a $(D+1)$-dimensional Euclidean spacetime, with a mass depending on $\lambda$. We now set up a PI starting from the action $A$. First, we sum over the orbital fluctuations at a fixed $\rho(\lambda)$. To find the correct measure of integration, we use the canonical formulation in which the action reads

$$A = \int_{\lambda_a}^{\lambda_b} \mathrm{d}\lambda \left[-\mathrm{i}px' + \frac{\rho(\lambda)p^2}{2M} + \frac{Mc^2}{2}\rho(\lambda)\right]. \tag{7}$$

After $\lambda$-slicing, the sliced action is given by

$$A^N = \sum_{n=1}^{N+1} \left[ -\mathrm{i} p_n (x_n - x_{n-1}) + \epsilon_n \rho_n \frac{p_n^2}{2M} + \frac{Mc^2}{2} \epsilon_n \rho_n \right]. \tag{8}$$

The measure of path integration now has the universal form

$$\int \mathrm{D}^{D+1} x \int \frac{\mathrm{D}^{D+1} p}{2\pi\hbar} = \prod_{n=1}^{N} \left[ \int \mathrm{d}^{D+1} x_n \right] \prod_{n=1}^{N+1} \left[ \frac{\mathrm{d}^{D+1} p_n}{(2\pi\hbar)^{D+1}} \right]. \tag{9}$$

The momenta are integrated out to give (setting $\lambda_{N+1} \equiv \lambda_b$, $\rho_{N+1} \equiv \rho_b$)

$$\frac{1}{\sqrt{2\pi\hbar\epsilon_b\rho_b/M}^{D+1}} \prod_{n=1}^{N} \left[ \int \frac{\mathrm{d}^{D+1} x_n}{\sqrt{2\pi\hbar\epsilon_n\rho_n/M}^{D+1}} \right] \exp\left\{ -\frac{1}{\hbar} A^N \right\} \tag{10}$$

with the time-sliced action in configuration space

$$A^N = \sum_{n=1}^{N+1} \left[ \frac{M}{2\epsilon_n\rho_n} (\triangle x_n)^2 + \frac{Mc^2}{2} \epsilon_n \rho_n \right]. \tag{11}$$

The Gaussian integrals over $x_n$ in equation (10) can now be done successively and we find

$$\frac{1}{\sqrt{2\pi\hbar L/Mc}^{D+1}} \exp\left[ -\frac{Mc}{2\hbar} \frac{(x_b - x_a)^2}{L} - \frac{Mc}{2\hbar} L \right] \tag{12}$$

where $L$ is the total sliced length of the orbit

$$L = c \sum_{n=1}^{N+1} \epsilon_n \rho_n \tag{13}$$

whose continuum limit is the total invariant length of the path

$$L = c \int_{\lambda_a}^{\lambda b} \mathrm{d}\lambda\, \rho(\lambda). \tag{14}$$

Remarkably, the result in equation (12) does not depend on the function $\rho(\lambda)$ but only on $L$. This is a reflection of the reparametrization invariance of the path integral. While the $\lambda$-interval changes under the transformation, the total length $L$ is invariant under the joint transformations in equations (5) and (6). This invariance permits only the invariant length $L$ to appear in the integrated expression of equation (12), and the PI over $\rho(\lambda)$ can be reduced to a simple integral over $L$. Therefore, the appropriate path integral for the time evolution amplitude reads

$$K(x_b, x_a) = \frac{\hbar}{2Mc} \int_0^\infty \mathrm{d}L \int \mathrm{D}\rho \Phi[\rho] \int \mathrm{d}^{D+1} x \,\mathrm{e}^{-A/\hbar} \tag{15}$$

where $\Phi[\rho]$ denotes a convenient gauge-fixing functional, for instance $\Phi[\rho] = \delta[\rho - 1]$ which fixes $\rho(\lambda)$ to unity everywhere. The solution of this integral is given by

$$K(x_b, x_a) = \frac{\hbar}{2Mc} \int_0^\infty \mathrm{d}L \frac{1}{\sqrt{2\pi\hbar L/Mc}^{D+1}} \exp\left[ -\frac{Mc}{2\hbar} \frac{(x_b - x_a)^2}{L} - \frac{Mc}{2\hbar} L \right]. \tag{16}$$

By Fourier transforming the $x$-dependence, this amplitude can also be written as

$$\begin{aligned} K(x_b, x_a) &= \frac{\hbar}{2Mc} \int_0^\infty \mathrm{d}L\, \mathrm{e}^{-McL/2\hbar} \int \frac{\mathrm{d}^{D+1} k}{(2\pi)^{D+1}} \exp\left[ \mathrm{i}k(x_b - x_a) - \frac{\hbar k^2}{2Mc} L \right] \\ &= \int \frac{\mathrm{d}^{D+1} k}{(2\pi)^{D+1}} \frac{1}{k^2 + M^2c^2/\hbar^2} \mathrm{e}^{\mathrm{i}k(x_b - x_a)}. \end{aligned} \tag{17}$$

This becomes the standard Green's function of the Klein–Gordon field

$$(-\partial_b^2 + M^2c^2/\hbar^2)K(x_b, x_a) = \delta^{(D+1)}(x_b - x_a). \tag{18}$$

The fixed-energy amplitude is related to equation (15) by a Laplace transformation:

$$G(x_b, x_a; E) = \mathrm{i}\int_{\tau_a}^{\infty} \mathrm{d}\tau_b\, \mathrm{e}^{E(\tau_b-\tau_a)} K(x_b, x_a). \tag{19}$$

Its poles and its cut along the energy axis contain information on the bound and continuous eigenstates of the system. The fixed-energy amplitude in $(D+1)$-dimensional Minkowski space with the time component $t = -\mathrm{i}\tau = -\mathrm{i}x^4/c$ has the following path integral representation [6]

$$G(\boldsymbol{x}_b, \boldsymbol{x}_a; E) = \frac{\mathrm{i}\hbar}{2Mc}\int_0^{\infty} \mathrm{d}L \int \mathrm{D}\rho\Phi[\rho]\int \mathrm{D}^D x\, \mathrm{e}^{-A_E/\hbar} \tag{20}$$

with the action integral

$$A_E = \int_{\lambda_a}^{\lambda_b} \mathrm{d}\lambda \left[\frac{M}{2\rho(\lambda)}\boldsymbol{x}'^2(\lambda) - \rho(\lambda)\frac{E^2}{2Mc^2} + \rho(\lambda)\frac{Mc^2}{2}\right]. \tag{21}$$

To prove this, we write the $x^{D+1}$ part of the sliced $(D+1)$-dimensional action of equation (11) in the canonical form of equation (8). Then we integrate out all $x_n^{D+1}$ variables, producing $n$ $\delta$-functions. These remove the integrals over the momentum variable $p_n^{D+1}$, leaving only a single integral over a common $p^{D+1}$. The Laplace transform of equation (19), finally, removes also this integral making $p^{D+1}$ equal to $-\mathrm{i}E/c$. In the continuum limit, we thus obtain the action of equation (21).

The path integral in equation (20) forms the basis to study relativistic particles in an external time-independent potential $V(\boldsymbol{x})$ by simply substituting the energy $E$ by $E - V(\boldsymbol{x})$.

## 3. The relativistic Wood–Saxon system and the relativistic step and square well potential problems

We now consider the Wood–Saxon potential described by (e.g. [8])

$$V(x) = -\frac{V_0}{1+\mathrm{e}^{(x-b)/R}}. \tag{22}$$

With this potential, equation (20) becomes

$$G(x_b, x_a; E) = \frac{\mathrm{i}\hbar}{2Mc}\int_0^{\infty} \mathrm{d}L \int \mathrm{D}\rho\, \Phi[\rho]\int \mathrm{D}x\,(\lambda) \exp\left\{-\frac{1}{\hbar}\int_{\lambda_a}^{\lambda_b} \mathrm{d}\lambda \left[\frac{M}{2\rho(\lambda)}x'^2 \right.\right.$$
$$\left.\left. -\frac{\rho(\lambda)}{2Mc^2}\left(E + \frac{V_0}{1+\mathrm{e}^{(x-b)/R}}\right)^2 + \rho(\lambda)\frac{Mc^2}{2}\right]\right\}. \tag{23}$$

This PI for $x(\lambda)$ is equivalent to the non-relativistic general Rosen–Morse one [10]. If we choose the gauge-fixing functional $\Phi[\rho]$ as an $\delta$-functional $\delta[\rho - 1]$, the solution is obtained as [1, 10]

$$G(x_b, x_a; E) = \mathrm{i}\frac{R}{c}\frac{\Gamma(m_1 - s)\Gamma(1 + m_1 + s)}{\Gamma(1 + m_1 + m_2)\Gamma(1 + m_1 - m_2)}$$
$$\times \left[\frac{1 - \tanh(\frac{x_b - b}{2R})}{2}\right]^{(m_1-m_2)/2}\left[\frac{1 + \tanh(\frac{x_b - b}{2R})}{2}\right]^{(m_1+m_2)/2}$$

$$\times \left[\frac{1-\tanh(\frac{x_a-b}{2R})}{2}\right]^{(m_1-m_2)/2}\left[\frac{1+\tanh(\frac{x_a-b}{2R})}{2}\right]^{(m_1+m_2)/2}$$
$$\times {}_2F_1\left[m_1-s, 1+m_1+s; 1+m_1-m_2; \frac{1-\tanh(\frac{x_>-b}{2R})}{2}\right]$$
$$\times {}_2F_1\left[m_1-s, 1+m_1+s; 1+m_1+m_2; \frac{1+\tanh(\frac{x_<-b}{2R})}{2}\right] \qquad (24)$$

with $x_{>,<}$ being the larger or smaller of $x_a, x_b$, and $b, R, V_0(>0)$ being constants, respectively. The energy is contained in the parameters

$$m_{1,2} = \frac{R}{\hbar c}\left[\sqrt{M^2c^4-(E+V_0)^2} \pm \sqrt{M^2c^4-E^2}\right] \qquad (25)$$

$$s = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1-4(RV_0/\hbar c)^2}. \qquad (26)$$

In the limit of $R \to 0$, the Wood–Saxon potential turns into the step potential $V^{(\mathrm{sp})}(x) = [\Theta(x-b)-1]V_0$ with the step height $V_0$. By taking the limit $R \to 0$ and inserting the asymptotic properties of the hypergeometry function

$${}_2F_1(a,b;c;z\approx 0) = 1 + z\frac{ab}{c} + \mathrm{O}(z^2) \qquad (27)$$

$${}_2F_1(a,b;c;z\approx 1) = \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-b)\Gamma(c-a)}$$
$$+(z-1)\frac{ab}{c}\frac{\Gamma(c+1)\Gamma(c-b-a-1)}{\Gamma(c-a)\Gamma(c-b)} + \mathrm{O}[(z-1)^2] \qquad (28)$$

into equation (24), the relativistic fixed-energy amplitude of the step potential is found to be

$$G^{(\mathrm{sp})}(x_b,x_a;E) = \Theta(b-x_b)\Theta(b-x_a)\frac{\mathrm{i}\hbar}{2Mc}\frac{Mc}{\sqrt{M^2c^4-(E+V_0)^2}}$$
$$\times \mathrm{e}^{-\mathrm{i}k(x_<-b)}\left[\mathrm{e}^{\mathrm{i}k(x_>-b)} - \frac{\chi+\mathrm{i}k}{\chi-\mathrm{i}k}\mathrm{e}^{-\mathrm{i}k(x_>-b)}\right]$$
$$+\Theta(x_b-b)\Theta(x_a-b)\frac{\mathrm{i}\hbar}{2Mc}\frac{Mc}{\sqrt{M^2c^4-E^2}}$$
$$\times \mathrm{e}^{-\chi(x_>-b)}\left[\mathrm{e}^{\chi(x_<-b)} + \frac{\chi+\mathrm{i}k}{\chi-\mathrm{i}k}\mathrm{e}^{-\chi(x_<-b)}\right] + \Theta(x_>-b)\Theta(b-x_<)\frac{\mathrm{i}\hbar}{2Mc}$$
$$\times \frac{2Mc}{\sqrt{M^2c^4-(E+V_0)^2}+\sqrt{M^2c^4-E^2}}\mathrm{e}^{-\chi(x_>-b)}\mathrm{e}^{-\mathrm{i}k(x_<-b)} \qquad (29)$$

with $\chi = \sqrt{M^2c^4-E^2}/\hbar c$, $k = \sqrt{(E+V_0)^2-M^2c^4}/\hbar c$. The continuity and boundary conditions of the amplitude can easily be checked.

To solve the relativistic square well problem, we first introduce an auxiliary $\delta$-function term into the action of equation (20) to form an impenetrable wall [12]. Then the fixed-energy amplitude in equation (20) becomes

$$G(x_b,x_a;E) = \frac{\mathrm{i}\hbar}{2Mc}\int_0^\infty \mathrm{d}L \int \mathrm{D}\rho\,\Phi[\rho]\mathrm{e}^{-\frac{1}{\hbar}\int_{\lambda_a}^{\lambda_b}\mathrm{d}\lambda\,\rho(\lambda)(\frac{Mc^2}{2}-\frac{E^2}{2Mc^2})}\int \mathrm{D}x\,\mathrm{e}^{-A_E/\hbar} \qquad (30)$$

with the auxiliary action

$$A_E = \int_{\lambda_a}^{\lambda_b}\mathrm{d}\lambda\left[\frac{M}{2\rho(\lambda)}x'^2(\lambda) - \rho(\lambda)\frac{(V^2-2VE)}{2Mc^2} + \rho(\lambda)\alpha\delta(x-a)\right]. \qquad (31)$$

Expanding the $\delta$-function part into the power series, we obtain the perturbative series [15, 16]

$$\begin{aligned} G(x_b, x_a; E) &= \frac{\mathrm{i}\hbar}{2Mc}\int_0^\infty \mathrm{d}L \int \mathrm{D}\rho\Phi[\rho]\mathrm{e}^{-\frac{1}{\hbar}\int_{\lambda_a}^{\lambda_b}\mathrm{d}\lambda\,\rho(\lambda)\left(\frac{Mc^2}{2}-\frac{E^2}{2Mc^2}\right)}\Bigg\{K_0(x_b, x_a; \lambda_b - \lambda_a) \\ &\quad + \sum_{n=1}^{\infty}\left(\frac{\alpha}{\hbar}\right)^n \frac{1}{n!}\prod_{i=1}^{n}\left[\int_{-\infty}^{\infty}\mathrm{d}x_i\,(\lambda)\int_{\lambda_a}^{\lambda_b}\mathrm{d}\lambda_i\,\rho(\lambda_i)\delta(x_i - a)\right] \\ &\quad \times K_0(x_1, x_a; \lambda_1)K_0(x_2, x_1; \lambda_2 - \lambda_1)\ldots K_0(x_n, x_{n-1}; \lambda_n - \lambda_{n-1}) \\ &\quad \times K_0(x_b, x_n; \lambda_b - \lambda_n)\Bigg\} \\ &= \frac{\mathrm{i}\hbar}{2Mc}\int_0^\infty \mathrm{d}L \int \mathrm{D}\rho\Phi[\rho]\mathrm{e}^{-\frac{1}{\hbar}\int_{\lambda_a}^{\lambda_b}\mathrm{d}\lambda\,\rho(\lambda)\left(\frac{Mc^2}{2}-\frac{E^2}{2Mc^2}\right)} \\ &\quad \times\Bigg\{K_0(x_b, x_a; \lambda_b - \lambda_a) + \sum_{n=1}^{\infty}\left(\frac{\alpha}{\hbar}\right)^n\int_{\lambda_a}^{\lambda_b}\mathrm{d}\lambda_n\,\rho(\lambda_n)\int_{\lambda_a}^{\lambda_n}\mathrm{d}\lambda_{n-1}\,\rho(\lambda_{n-1}) \\ &\quad \ldots\int_{\lambda_a}^{\lambda_2}\mathrm{d}\lambda_1\,\rho(\lambda_1)K_0(a, x_a; \lambda_1)K_0(a, a; \lambda_2 - \lambda_1) \\ &\quad \ldots K_0(a, a; \lambda_n - \lambda_{n-1})K_0(x_b, a; \lambda_b - \lambda_n)\Bigg\}. \end{aligned} \tag{32}$$

Here we have defined the $\lambda$-evolution amplitude

$$K_0(x_b, x_a; \lambda_b - \lambda_a) = \int \mathrm{D}x\,(\lambda)\mathrm{e}^{-\frac{1}{\hbar}\int_{\lambda_a}^{\lambda_b}\mathrm{d}\lambda\left[\frac{M}{2\rho(\lambda)}x'^2(\lambda)-\rho(\lambda)\frac{(V^2-2VE)}{2Mc^2}\right]} \tag{33}$$

and ordered the $\lambda$ as $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \lambda_b$ . We now choose $\Phi[\rho] = \delta[\rho - 1]$. The perturbative fixed-energy amplitude turns into

$$\begin{aligned} G(x_b, x_a; E) &= \frac{\mathrm{i}\hbar}{2Mc}\int_0^\infty \mathrm{d}L\,\mathrm{e}^{-\frac{1}{\hbar}\frac{L}{c}\left(\frac{Mc^2}{2}-\frac{E^2}{2Mc^2}\right)}\Bigg\{K_0(x_b, x_a; L) \\ &\quad + \sum_{n=1}^{\infty}\left(\frac{\alpha}{\hbar}\right)^n\int_{\lambda_a}^{\lambda_b}\mathrm{d}\lambda_n\int_{\lambda_a}^{\lambda_n}\mathrm{d}\lambda_{n-1}\ldots\int_{\lambda_a}^{\lambda_2}\mathrm{d}\lambda_1\,K_0(a, x_a; \lambda_1)K_0(a, a; \lambda_2 - \lambda_1) \\ &\quad \ldots K_0(a, a; \lambda_n - \lambda_{n-1})K_0(x_b, a; L/c - \lambda_n)\Bigg\}. \end{aligned} \tag{34}$$

We observe that the integration over invariant length $L$ is a Laplace transformation. Because of the convolution property of Laplace transformations, we see that

$$G(x_b, x_a; E) = G_0(x_b, x_a; E) - \frac{G_0(x_b, a; E)G_0(a, x_a; E)}{G_0(a, a; E) - \frac{\hbar}{\alpha}}. \tag{35}$$

The effect of an impenetrable wall appears at $x = a$ when we consider the limit $\alpha \to -\infty$ [12]. In this limit, we obtain the fixed-energy amplitude

$$G^{(\mathrm{wall})}(x_b, x_a; E) = G_0(x_b, x_a; E) - \frac{G_0(x_b, a; E)G_0(a, x_a; E)}{G_0(a, a; E)}. \tag{36}$$

Here it has been assumed that $G_0(a, a; E)$ actually exists. The similar trick in the non-relativistic potential problem was introduced in [12, 14, 17-19]. From equations (29) and (36), the relativistic fixed-energy amplitude of the square well potential in the half space $x > a$ is given by

$$G(x_b, x_a; E)^{(\mathrm{sw})} = \Theta(b - x_b)\Theta(b - x_a)\frac{\mathrm{i}\hbar}{2Mc}\frac{Mc}{\sqrt{M^2c^4 - (E + V_0)^2}}$$

$$
\begin{aligned}
&\times\left\{ e^{-ik(x_<-b)}\left[e^{ik(x_>-b)}-\frac{\chi+ik}{\chi-ik}e^{-ik(x_>-b)}\right]-\left[e^{2ik(a-b)}-\frac{\chi+ik}{\chi-ik}\right]^{-1}\right.\\
&\left.\times\left[e^{ik(x_b-b)}-\frac{\chi+ik}{\chi-ik}e^{-ik(x_b-b)}\right]\left[e^{ik(x_a-b)}-\frac{\chi+ik}{\chi-ik}e^{-ik(x_a-b)}\right]\right\}\\
&+\Theta(x_b-b)\Theta(x_a-b)\left\{\frac{i\hbar}{2Mc}\frac{Mc}{\sqrt{M^2c^4-E^2}}e^{-\chi(x_>-b)}\right.\\
&\times\left[e^{\chi(x_<-b)}+\frac{\chi+ik}{\chi-ik}e^{-\chi(x_<-b)}\right]\\
&-\frac{i\hbar}{2Mc}\frac{4Mc\sqrt{M^2c^4-(E+V_0)^2}}{\left[\sqrt{M^2c^4-(E+V_0)^2}+\sqrt{M^2c^4-E^2}\right]^2}\\
&\left.\times\left[e^{2ik(a-b)}-\frac{\chi+ik}{\chi-ik}\right]^{-1}e^{-\chi(x_b-b)}e^{-\chi(x_a-b)}\right\}\\
&+\Theta(x_>-b)\Theta(b-x_<)\frac{i\hbar}{2Mc}\frac{2Mc}{\sqrt{M^2c^4-(E+V_0)^2}+\sqrt{M^2c^4-E^2}}\\
&\times\left\{e^{-\chi(x_>-b)}e^{-ik(x_<-b)}-\left[e^{2ik(a-b)}-\frac{\chi+ik}{\chi-ik}\right]^{-1}\right.\\
&\left.\times\left[e^{ik(x_<-b)}-\frac{\chi+ik}{\chi-ik}e^{-ik(x_<-b)}\right]e^{-\chi(x_>-b)}\right\}.
\end{aligned}
\tag{37}
$$

Again, the continuity and boundary conditions are easily checked. The bound state energy [9] is determined by the poles of equation (37) and is given by

$$k/\chi=-\tan k(b-a). \tag{38}$$

Before closing this paper, we stress that by introducing the external steady Wood–Saxon potential the relativistic invariance is violated.

## 4. Conclusion

The fixed-energy amplitude of the relativistic Wood–Saxon potential systems has been solved by Kleinert's path integral approach to relativistic potential problems. From the solution, the fixed-energy amplitudes of the relativistic step and the square well potentials are also obtained by suitable limits and auxiliary $\delta$-function perturbation, respectively. As a physical application of equations (24) and (36), we make an impenetrable wall at $x=a$ for relativisitc particles which move in the Wood–Saxon potential. This model has been used to describe the interaction of a neutron with a heavy nucleus [8, 20]. The parameter $b$ is the nuclear radius, and the parameter $R$ determines the thickness of a surface layer in which the potential falls off from $V=0$ outside to $V=-V_0$ inside the nucleus.

With a spherically shaped $\delta$-function perturbation, there is a simple generalization for equation (36) if the external potentials are spherical symmetric. In this case, equation (36) becomes

$$G^{(\mathrm{wall})}(r_b,r_a;E)=G_0(r_b,r_a;E)-\frac{G_0(r_b,a;E)G_0(a,r_a;E)}{G_0(a,a;E)} \tag{39}$$

where the pure relativistic radial fixed-energy amplitude in $D$ dimensions is given by [1, 5]

$$G_0(r_b,r_a;E,l)=\frac{i\hbar}{2Mc}\int_0^\infty dL\int D\rho\Phi[\rho]\int Dr(\lambda)\exp\left\{-\frac{1}{\hbar}A_l[r,r']\right\} \tag{40}$$

with the action

$$A_l[r, r'] = \int_{\lambda_a}^{\lambda_b} \mathrm{d}\lambda \bigg[ \frac{M}{2\rho(\lambda)} r'^2(\lambda) + \frac{\rho(\lambda)\hbar^2}{2M} \frac{(l + D/2 - 1)^2 - \frac{1}{4}}{r^2} - \frac{\rho(\lambda)}{2Mc^2} [E - V(r)]^2 + \rho(\lambda) \frac{Mc^2}{2} \bigg]. \tag{41}$$

It is worth noting that we must require $a \neq 0$. From equation (41), one can easily discuss a relativistic spinless system with many constraints such as rings, radial boxes, etc.

We hope that the methods developed here are useful for obtaining other relativistic fixed-energy amplitudes by path integral.

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